WEEK 12:

Motivations: What perfectoid theory achieved:

Cohomology (Hodge....)

$X/\mathbb{Z}_p$ smooth projective, $T = \operatorname{Spec}\mathbb{Z}_p$. X is a family over T with 2 variables.
How does $H^i_{ét}(X_{\bar{t}}, \mathbb{F}_p)$ change?

Example: X elliptic curve,

$$\underbrace{H^1_{ét}(X_{\bar{\eta}}, \mathbb{F}_p)}_{\text{generic fiber}} \cong H^1_{sing}(X_{\mathbb{C}}, \mathbb{F}_p) \cong \mathbb{F}_p^2 \quad \text{but } H^1_{ét}(X_p, \mathbb{F}_p) = \begin{cases} 0 & \text{super sing.} \\ \mathbb{F}_p & \text{ordinary.} \end{cases}$$

Theorem: [Bhatt - Mossow (?) - Scholze] $H^*_{ét}(X_\eta, \mathbb{F}_p) \leftrightarrow H^*(X, \Omega^\bullet_{X/T})$.

Vanishings

Char 0: $H^i(X, \omega_X \otimes \mathcal{L}) = 0 \quad \forall i > 0$, X sin. proj., $\mathcal{L}$ ample. $\quad$ *char p: $H^i(X, \mathcal{L}^{-1}) \xrightarrow[0]{\exists f: X \xrightarrow{\text{finite}} Y} H^i(Y, f_*\mathcal{L}^{-1})$ for $i < \dim X$. (X proj., $\mathcal{L}$ big & s.ample)

Theorem: [Bhatt] "works" in mixed char (kills p-torsion)

Direct summand conj.

$A \overset{i}{\hookrightarrow} B$, B finite A-module $\Rightarrow$ i splits as an A-module hom.
(A regular) In char. 0 it is automatic, in char p ~ Kunz's thm.

Theorem [André] This holds.

Weight monodromy conj. (mixed char. version of Weil conj)

Weil conj.: eigenvalues of Frobenius on $H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ are alg. numbers of absolute value $q^{i/2}$

Theorem: [Scholze] Weight monodromy conjecture is true

Proposition: (A, A^+) complete affinoid, $a \in A$ is invertible $\Leftrightarrow$ $|a| \neq 0$ $\forall |\cdot| \in \operatorname{Spa}(A, A^+)$.

We prove a stronger version.

Theorem: (A, A^+) as above, $T \subseteq A$ and $J = T.A$, TFAE

1. $J = A$ $\quad$ 2. $\forall |\cdot| \in \operatorname{Spa}(A, A^+)$ $\exists t \in T$: $|t| \neq 0$
   ($\Leftrightarrow$ for T finite then $\{R(T/t) \mid t \in T\}$ open cover)

Proof: 1. $\Rightarrow$ 2. $\quad 1 = \sum_{\text{finite}} a_i t_i \quad a_i \in A \quad t_i \in T$

$1 = |1| = |\sum a_i t_i| \leq \max |a_i| . |t_i| \neq 0 \Rightarrow \exists i$ w/ $|t_i| \neq 0$.

2. $\Rightarrow$ 1. Suppose $J \neq A$ $\Rightarrow$ $\exists \mathfrak{m}$ maximal ideal such that $J \subseteq \mathfrak{m} \subseteq A$

Lemma: If A is a complete f-adic ring
(i) $A^\times$ is open (ii) $\mathfrak{m} \subseteq A$ max. ideal closed.

Proof: (i) $\frac{1}{1-a} = \sum_{i=0}^\infty a^i$ if $a \in A^{\circ\circ}$ $\Rightarrow$ $1 + A^{\circ\circ} \subseteq_{open} A^\times$

Thus for $a \in A^\times$, $a(1 + A^{\circ\circ}) \subseteq_{open} A^\times$ & $a \in a.(1 + A^{\circ\circ})$

(ii) $\mathfrak{m} \subseteq \underbrace{\bar{\mathfrak{m}}}_{\text{closure}} \subseteq A$ by (i) $\bar{\mathfrak{m}} \cap A^\times = \emptyset \Rightarrow \mathfrak{m} = \bar{\mathfrak{m}}$

Homework 17: (A, A^+) affinoid, $J \subseteq A$ ideal
then $\operatorname{Spa}(A/J, \underbrace{(A/J)^+}_{\text{int. closure}}) \cong \operatorname{Spa}(A, A^+) \cap \operatorname{Supp}^{-1}(J)$

In particular is $(A/J, (A/J)^+)$ is affinoid?

Enough to show that $\exists |\cdot| \in \operatorname{Spa}(A, A^+)$ st $\mathfrak{m} \subseteq \operatorname{Supp}|\cdot|$
$\overset{HW17}{\Longleftrightarrow} \operatorname{Spa}(\underbrace{A/\mathfrak{m}}_{\neq 0}, (A/\mathfrak{m})^+) \neq \emptyset$. We know that $0 \in A/\mathfrak{m}$ is closed
$\Leftrightarrow$ $A/\mathfrak{m}$ Hausdorff.

Proposition: (A, A^+) affinoid, TFAE
(i) $\mathrm{Spa}(A, A^+) = \emptyset$ (ii) $\mathrm{Cont}(A) = \emptyset$ (iii) $\overline{\{0\}} = A$

Proof: (i) $\Leftrightarrow$ (ii) is just density of $\mathrm{Spa}(A, A^+) \subseteq \mathrm{Cont}(A)$
(iii) $\Rightarrow$ (ii) $|\cdot| \in \mathrm{Cont}(A)$: $\mathrm{Supp}_{|\cdot|} = \{a \in A \mid |a| = 0\}$ is closed in $A \Rightarrow \mathrm{Supp}\,|\cdot| = A$ which is a contradiction $\Rightarrow \mathrm{Cont}(A) = \emptyset$.
(ii) $\Rightarrow$ (iii) $\overline{\{0\}} = \bigcap_{n \geq 0} I^n$ where (A_0, I) is a couple of def.
if $I^n = I^{n+1}$ for all $n \gg 0$ then $\{0\}$ is also open $\Rightarrow A/\overline{\{0\}}$ is discrete and every valuation is cont
$\xRightarrow{\mathrm{Cont}(A) = 0} \overline{\{0\}} = A$.

Claim: $p \subseteq q \subseteq A_0$ prime ideals if $I \subseteq q \Rightarrow I \subseteq p$. (this holds under the assumption (ii))

By Claim: $S = 1 + I$, $B = S^{-1}A_0$, $\varphi : A_0 \to B$, $m \subseteq B$ max. ideal, if $I \not\subseteq m \Rightarrow$
$B \to B/m$ so 1 has a preimage in $I.B$ of the form $\frac{a}{1+a'}$ then $1 - \frac{a}{1+a'}$ maps to 0
$I.B \longmapsto B/m$ but its invertible thus $I \subseteq m$.
$\Rightarrow \varphi(I) \subseteq \mathrm{Jac}(B) \xRightarrow{\text{Claim}} \varphi(I).B$ nilpotent $\xRightarrow{\varphi(I)B \text{ fin.gen.}} \exists a \in I : (1+a)I^n = 0$
$\Rightarrow I^n = I^{n+1}$.

Proof of the Claim: Suppose $I \not\subseteq p$, $(A_0/p)_{q/p} \subseteq \mathrm{Frac}(A_0/p) \Rightarrow$ valuation $|\cdot|$ on A_0 such that $\mathrm{Supp}|\cdot| = p$ & $\{a \in A_0 \mid |a| < 1\} = q \Rightarrow |\cdot|$ is analytic (i.e. $\mathrm{Supp}|\cdot|$ not open).

Questions: ① Is $|\cdot|$ continuous ② Can it be extended to A ?

Lemma. $B \subseteq A$ open subring of a topological ring. Then the map induced by restriction $\mathrm{Cont}(A)_{an} \twoheadrightarrow \mathrm{Cont}(B)_{an}$ is surjective.

This answers ②. We are left to show that $|\cdot|$ can be deformed to a continuous valuation keeping it analytic.

Problem, if $|I| \cap c\Gamma = \emptyset$. Define $c\Gamma(I) = \begin{cases} c\Gamma & \text{if } |I| \cap c\Gamma \neq \emptyset \\ \{\gamma \in \Gamma \mid \exists n > 0 : \delta^n \leq \gamma \leq \delta^{-n}\} & \text{if } |I| \cap c\Gamma = \emptyset \end{cases}$
where $\delta = \max\{|I|\}$

Why is $|\cdot|_{c\Gamma(I)}$ continuous if $|I| \cap c\Gamma = \emptyset$. Enough to show that $\{a \in A_0 \mid |a| < \delta^n\}$ is open for every n. This is because δ generates $c\Gamma(I)$ as a convex subgroup of Γ. But $I^{n+1} \subseteq \{a \in A_0 \mid |a| < \delta^n\}$.